MATH 1700: SECTION 12.2: THE LAW OF COSINES

In the previous section, we explored the Law of Sines. In order to get started using the Law of Sines, we needed to have at least one angle-side pair to get started. In this section, we present the Law of Cosines which can be used in the cases where the Law of Sines cannot.

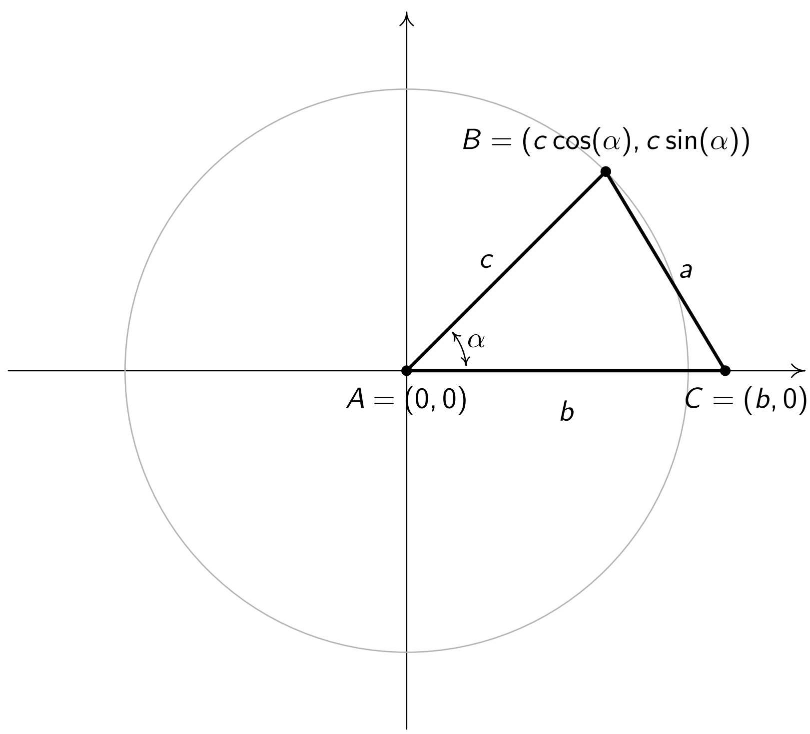
THE LAW OF COSINES: Given a triangle with angle-side opposite pairs (α, a), (β, b) and (γ, c), the following equations hold

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha) \qquad b^2 = a^2 + c^2 - 2ac\cos(\beta) \qquad c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

or, solving for the cosine in each equation, we have

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos(\beta) = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}$$

To prove the theorem, we consider a generic triangle with the vertex of angle α at the origin with side b positioned along the positive x-axis as sketched in the diagram below.



From this set-up, we immediately find that the coordinates of A and C are $A(0, 0)$ and $C(b, 0)$. Since the point $B(x, y)$ lies on a circle of radius c, the coordinates of B are $B(x, y) = B(c\cos(\alpha), c\sin(\alpha))$. (This would be true even if α were an obtuse or right angle so although we have drawn the case when α is acute, the following computations hold for any angle α drawn in standard position where $0 < \alpha < 180°$.)

We note that the distance between the points B and C is none other than the length of side a!

Using the distance formula (a.k.a. the Pythagorean Theorem!) we get:

$$
\begin{aligned}
a &= \sqrt{(c\cos(\alpha)-b)^2+(c\sin(\alpha)-0)^2} \\
a^2 &= \left(\sqrt{(c\cos(\alpha)-b)^2+c^2\sin^2(\alpha)}\right)^2 \\
a^2 &= (c\cos(\alpha)-b)^2+c^2\sin^2(\alpha) \\
a^2 &= c^2\cos^2(\alpha)-2bc\cos(\alpha)+b^2+c^2\sin^2(\alpha) \\
a^2 &= c^2\left(\cos^2(\alpha)+\sin^2(\alpha)\right)+b^2-2bc\cos(\alpha) \\
a^2 &= c^2(1)+b^2-2bc\cos(\alpha) && \text{Since } \cos^2(\alpha)+\sin^2(\alpha)=1 \\
a^2 &= c^2+b^2-2bc\cos(\alpha)
\end{aligned}
$$

The remaining formulas can be derived by simply reorienting the triangle to place a different vertex at the origin.

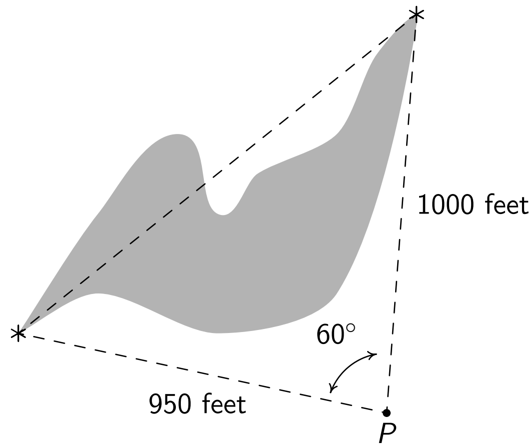
What's important about a and α in the above proof is that (α, a) is an angle-side opposite pair and b and c are the sides adjacent to α – the same can be said of any other angle-side opposite pair in the triangle.

EXAMPLE 1: Solve the following triangles. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

1. $\beta = 50°$, $a = 7$ units, $c = 2$ units

2. $a = 4$ units, $b = 7$ units, $c = 5$ units

EXAMPLE 2: A researcher wishes to determine the width of a vernal pond as drawn below. From a point P the distance to the eastern-most point of the pond is 950 feet, while the distance to the western-most point of the pond from P is 1000 feet. If the angle between the two lines of sight is 60°, find the width of the pond.



HERON'S FORMULA: Suppose a, b and c denote the lengths of the three sides of a triangle. Let s be the semiperimeter of the triangle, that is, let $s = \frac{1}{2}(a+b+c)$. Then the area A enclosed by the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

The proof of Heron's Formula is in the text and essentially combines the formula we derived for area in the previous section with the Law of Cosines.

EXAMPLE 3: Find the area enclosed by the triangle with side lengths $a = 4$ units, $b = 7$ units, and $c = 5$ units.

HOMEWORK: Section 12.2: Page 1045: 1-25 odd